Existence of closed geodesics on Finsler n-spheres

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Abstract

In this paper, we prove that on every Finsler n-sphere (S^n, F) with reversibility λ satisfying $F^2 < (\frac{\lambda+1}{\lambda})^2 g_0$ and $l(S^n, F) \geq \pi(1 + \frac{1}{\lambda})$, there always exist at least n prime closed geodesics without self-intersections, where g_0 is the standard Riemannian metric on S^n with constant curvature 1 and $l(S^n, F)$ is the length of a shortest geodesic loop on (S^n, F). We also study the stability of these closed geodesics.

Key words: Finsler spheres, closed geodesics, equivariant Morse theory.

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Running head: Closed geodesics on Finsler spheres

1 Introduction and main results

There is a famous conjecture in Riemannian geometry which claims there exist infinitely many prime closed geodesics on any Riemannian manifold. This conjecture has been proved except for CROSS's (compact rank one symmetric spaces). The results of J. Franks [Fra] in 1992 and V. Bangert [Ban] in 1993 imply this conjecture is true for any Riemannian 2-sphere. But once one move to the Finsler case, the conjecture becomes false. It was quite surprising when A. Katok [Kat] in 1973 found some non-symmetric Finsler metrics on CROSS's with only finitely many prime closed geodesics and all closed geodesics are non-degenerate and elliptic. In Katok's examples the spheres

S^{2n} and S^{2n-1} have precisely $2n$ closed geodesics. Based on Katok's work, D. V. Anosov [Ano] in 1974's ICM conjectured that there exist at least $2[\frac{n+1}{2}]$ prime closed geodesics on any Finsler n-sphere (S^n, F). In [Zil], W. Ziller conjectured the number of prime closed geodesics on any Finsler n-sphere (S^n, F) is at least n. This paper is devoted to a study on the number prime closed geodesics on Finsler n-spheres. Let us recall firstly the definition of the Finsler metrics.

Definition 1.1. (cf. [She]) *Let M be a finite dimensional smooth manifold. A function $F : TM \to [0, +\infty)$ is a* Finsler metric *if it satisfies*

(F1) F is C^∞ on $TM \setminus \{0\}$,

(F2) $F(x, \lambda y) = \lambda F(x, y)$ for all $x \in M$, $y \in T_xM$ and $\lambda > 0$,

(F3) For every $y \in T_xM \setminus \{0\}$, the quadratic form

$$g_{x,y}(u,v) \equiv \frac{1}{2} \frac{\partial^2}{\partial s \partial t} F^2(x, y + su + tv)|_{t=s=0}, \qquad \forall u, v \in T_xM,$$

is positive definite.

In this case, (M, F) is called a Finsler manifold. *F is* symmetric *if $F(x, -y) = F(x, y)$ holds for all $x \in M$ and $y \in T_xM$. F is* Riemannian *if $F(x, y)^2 = \frac{1}{2}G(x)y \cdot y$ for some symmetric positive definite matrix function $G(x) \in GL(T_xM)$ depending on $x \in M$ smoothly.*

A closed curve in a Finsler manifold is a closed geodesic if it is locally the shortest path connecting any two nearby points on this curve (cf. [She]). As usual, on any Finsler manifold (M, F), a closed geodesic $c : S^1 = \mathbf{R}/\mathbf{Z} \to M$ is *prime* if it is not a multiple covering (i.e., iteration) of any other closed geodesics. Here the m-th iteration c^m of c is defined by $c^m(t) = c(mt)$. The inverse curve c^{-1} of c is defined by $c^{-1}(t) = c(1-t)$ for $t \in \mathbf{R}$. Note that unlike Riemannian manifold, the inverse curve c^{-1} of a closed geodesic c on a non-symmetric Finsler manifold need not be a geodesic. We call two prime closed geodesics c and d *distinct* if there is no $\theta \in (0, 1)$ such that $c(t) = d(t + \theta)$ for all $t \in \mathbf{R}$. We shall omit the word *distinct* when we talk about more than one prime closed geodesic. On a symmetric Finsler (or Riemannian) manifold, two closed geodesics c and d are called *geometrically distinct* if $c(S^1) \neq d(S^1)$, i.e., their image sets in M are distinct.

For a closed geodesic c on (M, F), denote by P_c the linearized Poincaré map of c (cf. p.143 of [Zil]). Then $P_c \in \mathrm{Sp}(2n - 2)$ is a symplectic matrix. For any $M \in \mathrm{Sp}(2k)$, we define the *elliptic height* $e(M)$ of M to be the total algebraic multiplicity of all eigenvalues of M on the unit circle $\mathbf{U} = \{z \in \mathbf{C}| \ |z| = 1\}$ in the complex plane $\mathbf{C}$. Since M is symplectic, $e(M)$ is even and $0 \leq e(M) \leq 2k$. Then c is called *hyperbolic* if all the eigenvalues of P_c avoid the unit circle in $\mathbf{C}$, i.e., $e(P_c) = 0$; *elliptic* if all the eigenvalues of P_c are on the unit circle, i.e., $e(P_c) = 2(n - 1)$.

Following H-B. Rademacher in [Rad4], the reversibility $\lambda = \lambda(M, F)$ of a compact Finsler manifold (M, F) is defied to be

$$\lambda := \max\{F(-X) \,|\, X \in TM, F(X) = 1\} \geq 1.$$

We are aware of a number of results concerning closed geodesics on spheres. According to the classical theorem of Lyusternik-Fet [LyF] from 1951, there exists at least one closed geodesic on every compact Riemannian manifold. The proof of this theorem is variational and carries over to the Finsler case. Motivated by the work [Kli1] of W. Klingenberg in 1969, W. Ballmann, G. Thorbergsson and W. Ziller studied in [BTZ1] and [BTZ2] of 1982-83 the existence and stability of closed geodesics on positively curved CROSS's. In [Hin] of 1984, N. Hingston proved that a Riemannian metric on a sphere all of whose closed geodesics are hyperbolic carries infinitely many geometrically distinct closed geodesics. By the results of J. Franks in [Fra] of 1992 and V. Bangert in [Ban] of 1993, there are infinitely many geometrically distinct closed geodesics for any Riemannian metric on S^2. In [Rad5], H.-B. Rademacher studied the existence and stability of closed geodesics on positively curved Finsler manifolds. In [BaL], V. Bangert and Y. Long proved that on any Finsler 2-sphere (S^2, F), there exist at least two prime closed geodesics. In [LoW] of Y. Long and the author, they further proved the existence of at least two irrationally elliptic prime closed geodesics on every Finsler 2-sphere (S^2, F) provided the number of prime closed geodesics is finite. In [Wang], the author proved there exist three prime closed geodesics on any (S^3, F) satisfying $(\frac{\lambda}{\lambda+1})^2 < K \leq 1$, where K is the flag curvature of (S^3, F).

The following are the main results in this paper:

Theorem 1.2. *On every Finsler n-sphere (S^n, F) with reversibility λ satisfying $F^2 < (\frac{\lambda+1}{\lambda})^2 g_0$ and $l(S^n, F) \geq \pi(1+\frac{1}{\lambda})$, there always exist at least n prime closed geodesics without self-intersections, where g_0 is the standard Riemannian metric on S^n with constant curvature 1 and $l(S^n, F)$ is the length of a shortest geodesic loop on (S^n, F).*

Theorem 1.3. *On every Finsler n-sphere (S^n, F) with reversibility λ satisfying $F^2 < (\frac{\lambda+1}{\lambda})^2 g_0$ and $\left(\frac{\lambda}{\lambda+1}\right)^2 < K \leq 1$ ($0 < K \leq 1$ if n is even), there always exist at least n prime closed geodesics without self-intersections, where g_0 is given by Theorem 1.2.*

Theorem 1.4. *On every Finsler n-sphere (S^n, F) with reversibility λ satisfying $F^2 < (\frac{2(n-1)}{2n-1})^2 g_0$ and $(\frac{2n-3}{n-1}\frac{\lambda}{\lambda+1})^2 < K \leq 1$, there exist at least $2[\frac{n}{2}]$ non-hyperbolic prime closed geodesics, where g_0 is given by Theorem 1.2.*

Remark 1.5. The proof of these theorems is motivated by [EL], [Rad3], [BTZ1] and [BTZ2]. We use the S^1-equivariant Morse theory to obtain n critical values of the energy functional E on

the space pair (Λ, Λ^0), where Λ is the free loop space of S^n and Λ^0 is its subspace consisting of constant point curves. Then we use the assumptions to show that these critical values correspond to n distinct prime closed geodesics. The stability results are obtained by considering the indices of iterations of these closed geodesics, cf. [Bot2]. Note that the methods in [BTZ1] and [BTZ2] can't be used for non-symmetric Finsler metrics because the lack of $\mathbf{Z}_2$-symmetry for E.

In this paper, let $\mathbf{N}$, $\mathbf{N}_0$, $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{C}$ denote the sets of natural integers, non-negative integers, integers, rational numbers, real numbers, and complex numbers respectively. We use only singular homology modules with $\mathbf{Q}$-coefficients. We denote by $[a] = \max\{k \in \mathbf{Z} \,|\, k \leq a\}$ for any $a \in \mathbf{R}$.

2 Critical point theory for closed geodesics

In this section, we will study critical point theory for closed geodesics.

On a compact Finsler manifold (M, F), we choose an auxiliary Riemannian metric. This endows the space $\Lambda = \Lambda M$ of H^1-maps $\gamma : S^1 \to M$ with a natural Riemannian Hilbert manifold structure on which the group $S^1 = \mathbf{R}/\mathbf{Z}$ acts continuously by isometries, cf. [Kli2], Chapters 1 and 2. This action is defined by translating the parameter, i.e.,

$$(s \cdot \gamma)(t) = \gamma(t + s)$$

for all $\gamma \in \Lambda$ and $s, t \in S^1$. The Finsler metric F defines an energy functional E and a length functional L on Λ by

$$E(\gamma) = \frac{1}{2} \int_{S^1} F(\dot{\gamma}(t))^2 dt, \quad L(\gamma) = \int_{S^1} F(\dot{\gamma}(t)) dt. \tag{2.1}$$

Both functionals are invariant under the S^1-action. By [Mer], the functional E is $C^{1,1}$ on Λ and satisfies the Palais-Smale condition. Thus we can apply the S^1-deformation theorems in [Cha] and [MaW]. The critical points of E of positive energies are precisely the closed geodesics $c : S^1 \to M$ of the Finsler structure. If $c \in \Lambda$ is a closed geodesic then c is a regular curve, i.e., $\dot{c}(t) \neq 0$ for all $t \in S^1$, and this implies that the formal second differential $E''(c)$ of E at c exists. As usual we define the index $i(c)$ of c as the maximal dimension of subspaces of $T_c\Lambda$ on which $E''(c)$ is negative definite, and the nullity $\nu(c)$ of c so that $\nu(c) + 1$ is the dimension of the null space of $E''(c)$. In fact, one can use a finite dimension approximation Δ of Λ as in [Rad2] such that $F|_\Delta$ is C^∞ and define the index and nullity, one can prove those equal to the above defined ones.

For $m \in \mathbf{N}$ we denote the m-fold iteration map $\phi^m : \Lambda \to \Lambda$ by

$$\phi^m(\gamma)(t) = \gamma(mt) \qquad \forall \gamma \in \Lambda, t \in S^1. \tag{2.2}$$

We also use the notation $\phi^m(\gamma) = \gamma^m$. For a closed geodesic c, the average index is defined by

$$\hat{i}(c) = \lim_{m \to \infty} \frac{i(c^m)}{m}. \tag{2.3}$$

If $\gamma \in \Lambda$ is not constant then the multiplicity $m(\gamma)$ of γ is the order of the isotropy group $\{s \in S^1 \mid s \cdot \gamma = \gamma\}$. If $m(\gamma) = 1$ then γ is called *prime*. Hence $m(\gamma) = m$ if and only if there exists a prime curve $\tilde{\gamma} \in \Lambda$ such that $\gamma = \tilde{\gamma}^m$.

In this paper for $\kappa \in \mathbf{R}$ we denote by

$$\Lambda^\kappa = \{d \in \Lambda \mid E(d) \leq \kappa\}, \tag{2.4}$$

For a closed geodesic c we set

$$\Lambda(c) = \{\gamma \in \Lambda \mid E(\gamma) \leq E(c)\}.$$

We have the following property for the space pair $H_{S^1,*}(\Lambda S^n, \Lambda^0 S^n)$, where $H_{S^1,*}$ is the S^1-equivariant homology in the sense of A. Borel, cf. [Bor].

Proposition 2.1. *There are n subordinate nonzero homology classes $\sigma_k \in H_{S^1,n-1+2k}(\Lambda S^n, \Lambda^0 S^n)$ for $0 \leq k \leq n - 1$.*

Proof. Let g_0 be the standard metric on S^n with constant curvature 1 and E_0 be the corresponding energy functional. Then by §4 of [Hin], $H_{S^1,*}(\Lambda S^n, \Lambda^0 S^n)$ is generated by the local critical groups

$$H_{S^1,*}(\Lambda_0^{2m^2\pi^2+\epsilon} S^n, \Lambda_0^{2m^2\pi^2-\epsilon} S^n) \simeq H_{S^1,*-(2m-1)(n-1)}(T_1 S^n), \quad m \in \mathbf{N} \tag{2.5}$$

where $\Lambda_0^\kappa = \{d \in \Lambda \mid E_0(d) \leq \kappa\}$, $T_1 S^n = \{v \in TS^n \mid |v|_{g_0} = 1\}$ and $\epsilon > 0$ is sufficiently small. In fact, this follows since the Morse series is lacunary, thus the energy functional E_0 is perfect. Now we consider $H_{S^1,*}(T_1 S^n)$. Note that the isotropy group of the S^1-action on $T_1 S^n$ is $\mathbf{Z}_m$ for $m \in \mathbf{N}$. Thus it follows from Lemma 6.11 of [FaR] and the universal coefficient theorem that

$$H_{S^1,*}(T_1 S^n) \simeq H_*(T_1 S^n / S^1). \tag{2.6}$$

Now consider the case $m = 1$. In this case we have an S^1-fibration $S^1 \to T_1 S^n \to T_1 S^n / S^1$. Since the S^1-action is free, $T_1 S^n / S^1$ is a smooth manifold. Let $e \in H^2(T_1 S^n / S^1)$ be the Euler class of the S^1-fibration $T_1 S^n \to T_1 S^n / S^1$. Then it follows from [Wei] that $e^{n-1} \neq 0$ is a generator of $H^{2n-2}(T_1 S^n / S^1)$. Denote by $[C]$ the fundamental class of $T_1 S^n / S^1$. Then we have n subordinate nonzero homology classes $\alpha_k \in H_{S^1,2k}(T_1 S^n / S^1)$ for $0 \leq k \leq n - 1$ defined by $\alpha_k = [C] \cap e^{n-1-k}$. Now we construct σ_k from α_k as the following. First note that $T_1 S^n / S^1$ is a non-degenerate critical

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S^1-manifold in the sense of [Bot1]. Thus it follows from the handle-bundle theorem in [Was], $\Lambda_0^{2\pi^2+\epsilon} S^n$ is S^1-homotopic to $\Lambda_0^{2\pi^2-\epsilon} S^n$ with the handle-bundle DN^- attached along SN^-, where DN^- is the closed disk bundle of the negative bundle over $T_1 S^n / S^1$ and $SN^- = \partial DN^-$. In particular, rank$DN^- = n-1$. Now we have

$$H_{S^1,*}(\Lambda_0^{2\pi^2+\epsilon} S^n, \Lambda_0^{2\pi^2-\epsilon} S^n) \simeq H_{S^1,*}(DN^-, SN^-) \simeq H_{S^1,*-(n-1)}(T_1 S^n). \tag{2.7}$$

The latter isomorphism is given by the Thom isomorphism Φ. Let $f : (\Lambda S^n)_{S^1} \equiv \Lambda S^n \times_{S^1} S^\infty \to \mathbf{CP}^\infty$ be a classifying map and $\eta \in H^2(\mathbf{CP}^\infty)$ be the universal first rational Chern class. Let $\sigma_k = \Phi^{-1}(\alpha_k) = \Phi^{-1}([C] \cap e^{n-1-k}) \neq 0$. Then clearly we have $\sigma_k = \Phi^{-1}([C]) \cap (f^* \eta)^{n-1-k}$. In fact, this follows since the Euler class $e \in H^2(T_1 S^n)$ coincide with the first Chern class $c_1 \in H^2(T_1 S^n)$ of the S^1-bundle $T_1 S^n \to T_1 S^n / S^1$; the S^1-action on $T_1 S^n$ is free, thus $(T_1 S^n)_{S^1}$ is homotopic to $T_1 S^n / S^1$; and DN^- is S^1-homotopic to $T_1 S^n$.

The proof of the proposition is complete. ∎

Now suppose $\sigma_k \in H_{S^1, n-1+2k}(\Lambda S^n, \Lambda^0 S^n)$ for $0 \leq k \leq n-1$ are given by Proposition 2.1. Let $\lambda_k = \inf_{\gamma \in \sigma_k} \sup_{d \in \text{im}\gamma} E(d)$. Then we have the following

Proposition 2.2. *Each λ_k is a critical value of E and $0 < \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{n-1}$. In particular, if $\lambda_k = \lambda_{k+1}$ for some $0 \leq k < n-1$, then there are infinitely many prime closed geodesics on (S^n, F).*

Proof. By [Mer], the functional E is $C^{1,1}$ on Λ and satisfies the Palais-Smale condition. Thus the negative gradient flow of E exists and is S^1-equivariant. Suppose λ_k is not a critical value of E, then by the Palais-Smale condition, there exist some constants $\delta, \rho > 0$ such that $\|\text{grad}E\| \geq \delta$ for any $d \in E^{-1}[\lambda_k - \rho, \lambda_k + \rho]$. Now by definition of λ_k, we have a chain $\gamma \in \sigma_k$ such that $\sup_{d \in \text{im}\gamma} E(d) < \lambda_k + \rho$. Thus we can push γ by the negative gradient flow of E down the level $\lambda_k - \rho$ to obtain γ'. Since the flow is S^1-equivariant, we have $[\gamma] = [\gamma']$. This contradicts to the definition of λ_k.

Since any nonconstant geodesic loop c on (S^n, F) satisfies $L(c) > \varrho$ for some $\varrho > 0$. We have $E(c) \geq \frac{1}{2} L(c)^2 > \frac{1}{2} \varrho^2 > 0$ for any nonconstant closed geodesic c on (S^n, F). We claim that $\lambda_0 \geq \frac{1}{2} \varrho^2$. In fact, suppose $\lambda_0 < \frac{1}{2} \varrho^2$, then by definition of λ_0, we have a chain $\gamma \in \sigma_0$ such that $\sup_{d \in \text{im}\gamma} E(d) < \frac{1}{2} \varrho^2$. Since there is no critical value of E in the interval $(0, \frac{1}{2} \varrho^2)$, we have $\sigma_0 \in H_{S^1, n-1}(\Lambda^{\frac{1}{2} \varrho^2 - \epsilon} S^n, \Lambda^0 S^n) = 0$, where $\epsilon > 0$ is sufficiently small. This contradicts to $\sigma_0 \neq 0$.

The relation $\lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{n-1}$ is obvious by the definition of cap product.

Now suppose $\lambda_k = \lambda_{k+1}$ for some $0 \leq k < n-1$. Denote by $crit(E)$ the set of critical points of E. Suppose $\mathcal{U}$ is any S^1-invariant neighborhood of $E^{-1}(\lambda_k) \cap crit(E)$. Then by Theorem 1.7.1

of [Cha], there exist $\epsilon > 0$ and an S^1-equivariant deformation $F : [0,1] \times \Lambda^{\lambda_k+\epsilon} \to \Lambda^{\lambda_k+\epsilon}$ such that $F(1, \Lambda^{\lambda_k+\epsilon} \setminus \mathcal{U}) \subset \Lambda^{\lambda_k-\epsilon}$. In particular, F induces a deformation

$$\widetilde{F} : [0,1] \times (\Lambda^{\lambda_k+\epsilon})_{S^1} \to (\Lambda^{\lambda_k+\epsilon})_{S^1}, \quad \widetilde{F}(1, (\Lambda^{\lambda_k+\epsilon} \setminus \mathcal{U})_{S^1}) \subset (\Lambda^{\lambda_k-\epsilon})_{S^1}.$$

By definition of λ_{k+1}, we have a chain $\gamma \in \sigma_{k+1}$ such that $\sup_{d \in \text{im} \gamma} E(d) < \lambda_{k+1}+\epsilon = \lambda_k+\epsilon$. Choose a subdivision γ' of γ such that any simplex of γ' is either contained in $\mathcal{U}_{S^1}$ or in $(\Lambda^{\lambda_k+\epsilon} \setminus \mathcal{U})_{S^1}$. Note that by Proposition 2.1, we have $\sigma_k = [\gamma'] \cap \eta$ for some $\eta \in H^2_{S^1}(\Lambda S^n)$. Let $i : \mathcal{U}_{S^1} \to (\Lambda S^n)_{S^1}$ and $j : (\Lambda S^n)_{S^1} \to ((\Lambda S^n)_{S^1}, \mathcal{U}_{S^1})$ be the inclusions. If $i^*(\eta) = 0$, then we have $\eta = j^*(\theta)$ for some $\theta \in H^2_{S^1}(\Lambda S^n, \mathcal{U})$ by the exactness of the cohomology sequence of the space pair $((\Lambda S^n)_{S^1}, \mathcal{U}_{S^1})$. This implies there is a cocycle in η such that $\eta(\tau) = 0$ for any simplex $\tau \subset (\Lambda S^n)_{S^1}$ with support $|\tau| \subset \mathcal{U}_{S^1}$. Thus $\gamma' \cap \eta$ is a cycle in σ_k with support $|\gamma' \cap \eta| \subset (\Lambda^{\lambda_k+\epsilon} \setminus \mathcal{U})_{S^1}$. Now $[\widetilde{F}(1, \gamma' \cap \eta)] = [\gamma' \cap \eta]$ and $|\widetilde{F}(1, \gamma' \cap \eta)| \subset \Lambda^{\lambda_k-\epsilon}$. This contradicts to the definition of λ_k. Hence we must have $i^*(\eta) \neq 0$.

Now suppose there are finitely many prime closed geodesics on (S^n, F). Then we have $E^{-1}(\lambda_k) \cap \text{crit}(E)$ is a union of finite many disjoint circles. Thus we can choose $\mathcal{U}$ to be a union of finite many tubular neighborhoods of this circles. This implies $i^*(\eta) = 0$. This contradiction proves the proposition. ∎

We call a closed geodesic c *isolated*, if there exists an S^1-invariant neighborhood $\mathcal{N}$ of $S^1 \cdot v$ such that $\text{crit}(E) \cap \mathcal{N} = S^1 \cdot c$.

Proposition 2.3. *Suppose there exists $\delta_k > 0$ such that any closed geodesic c with $E(c) \in (\lambda_k - \delta_k, \lambda_k + \delta_k)$ is isolated. Then there exists a closed geodesic c_k such that*

$$E(c_k) = \lambda_k, \qquad i(c_k) \leq n - 1 + 2k \leq i(c_k) + \nu(c_k)$$

for $0 \leq k \leq n-1$.

Proof. By assumption and the Palais-Smale condition, we can choose $\epsilon > 0$ small enough such that λ_k is the unique critical value of E in $(\lambda_k - \epsilon, \lambda_k + \epsilon)$.

By definition of λ_k, we have a chain $\gamma \in \sigma_k$ such that $\sup_{d \in \text{im} \gamma} E(d) < \lambda_k + \epsilon$. Thus we have $0 \neq [\gamma] \in H_{S^1, n-1+2k}(\Lambda^{\lambda_k+\epsilon}, \Lambda^0)$. If $H_{S^1, n-1+2k}(\Lambda^{\lambda_k+\epsilon}, \Lambda^{\lambda_k-\epsilon}) = 0$, then by the exactness of the homology sequence of the triple $((\Lambda^{\lambda_k+\epsilon})_{S^1}, (\Lambda^{\lambda_k-\epsilon})_{S^1}, (\Lambda^0)_{S^1})$, we have $\gamma' \in H_{S^1, n-1+2k}(\Lambda^{\lambda_k-\epsilon}, \Lambda^0)$. such that $i_*(\gamma') = \gamma$, where $i : ((\Lambda^{\lambda_k-\epsilon})_{S^1}, (\Lambda^0)_{S^1}) \to ((\Lambda^{\lambda_k+\epsilon})_{S^1}, (\Lambda^0)_{S^1})$ is the inclusion. This contradicts to the definition of λ_k. Hence we have

$$H_{S^1, n-1+2k}(\Lambda^{\lambda_k+\epsilon}, \Lambda^{\lambda_k-\epsilon}) \neq 0. \tag{2.8}$$

Since λ_k is the unique critical value of E in $(\lambda_k - \epsilon, \lambda_k + \epsilon)$. By the equivariant version of Lemma 1.4.2 of [Cha], we have

$$H_{S^1,*}(\Lambda^{\lambda_k+\epsilon}, \Lambda^{\lambda_k-\epsilon}) \simeq \bigoplus_{c \in crit(E)} C_{S^1,*}(E, S^1 \cdot c), \tag{2.9}$$

where $C_{S^1,*}(E, S^1 \cdot c)$ is the S^1-critical group at $S^1 \cdot c$ defined by

$$C_{S^1,\,*}(E,\ S^1 \cdot c) = H_{S^1,\,*}(\Lambda(c) \cap \mathcal{N},\ (\Lambda(c) \setminus S^1 \cdot c) \cap \mathcal{N}), \tag{2.10}$$

where $\mathcal{N}$ is any S^1-invariant open neighborhood of $S^1 \cdot c$ such that $crit(E) \cap \mathcal{N} = S^1 \cdot c$. By shrinking $\mathcal{N}$ if necessary, we may assume the multiplicities $m(d)$ for $d \in \mathcal{N}$ is bounded from above. Then by Lemma 6.11 of [FaR], we have

$$H_{S^1}^*(\Lambda(c) \cap \mathcal{N},\ (\Lambda(c) \setminus S^1 \cdot c) \cap \mathcal{N}) \simeq H^*((\Lambda(c) \cap \mathcal{N})/S^1,\ ((\Lambda(c) \setminus S^1 \cdot c) \cap \mathcal{N})/S^1). \tag{2.11}$$

Now by introducing finite-dimensional approximations to Λ and apply Gromoll-Meyer theory as in §6 of [Rad2], we have

$$H^q((\Lambda(c) \cap \mathcal{N})/S^1,\ ((\Lambda(c) \setminus S^1 \cdot c) \cap \mathcal{N})/S^1) = 0 \tag{2.12}$$

provided $q > i(c) + \nu(c)$ or $q < i(c)$, cf. Satz 6.13 of [Rad2].

Now the proposition follows from (2.8)-(2.12). ∎

Proposition 2.4. *Suppose a closed geodesic c is non-degenerate, i.e., $\nu(c) = 0$, then c is isolated.*

Proof. Following [Rad2], Section 6.2, we introduce finite-dimensional approximations to Λ. We choose an arbitrary energy value $a > 0$ and $k \in \mathbf{N}$ such that every F-geodesic of length $< \sqrt{2a/k}$ is minimal. Then

$$\Lambda(k,a) = \left\{ \gamma \in \Lambda \mid E(\gamma) < a \text{ and } \gamma|_{[i/k,(i+1)/k]} \text{ is an } F\text{-geodesic for } i = 0, \ldots, k-1 \right\}$$

is a $(k \cdot \dim M)$-dimensional submanifold of Λ consisting of closed geodesic polygons with k vertices. The set $\Lambda(k,a)$ is invariant under the subgroup $\mathbf{Z}_k$ of S^1. Then $E|_{\Lambda(k,a)}$ is smooth. Closed geodesics in $\Lambda^{a-} = \{\gamma \in \Lambda \mid E(\gamma) < a\}$ are precisely the critical points of $E|_{\Lambda(k,a)}$, and for every closed geodesic $c \in \Lambda(k,a)$ the index of $(E|_{\Lambda(k,a)})''(c)$ equals $i(c)$ and the null space of $(E|_{\Lambda(k,a)})''(c)$ coincides with the null space of $E''(c)$, cf. [Rad2], p.51. Clearly, $S^1 \cdot c \subset \Lambda(k,a)$ is a critical manifold and its tangent space contains in the null space of $(E|_{\Lambda(k,a)})''(c)$. Since $\nu(c) = 0$ by assumption, we have $(E|_{\Lambda(k,a)})''(c)|_{N(S^1 \cdot c)}$ is non-degenerate. where $N(S^1 \cdot c)$ is the normal bundle of $S^1 \cdot c$ in $\Lambda(k,a)$. Thus $S^1 \cdot c$ is a non-degenerate critical manifold of $E|_{\Lambda(k,a)}$ in the sense of [Bot1]. Hence it is an isolated critical manifold of $E|_{\Lambda(k,a)}$, and then it must be a isolated critical manifold of E. ∎

3 Proof of the main theorems

In this section, we give the proofs of the main theorems.

Proof of Theorem 1.2. Suppose the Finsler n-sphere (S^n, F) with reversibility λ satisfying $F^2 < (\frac{\lambda+1}{\lambda})^2 g_0$ and $l(S^n, F) \geq \pi(1 + \frac{1}{\lambda})$. Then we have $\Lambda_0^{2\pi^2+\epsilon} S^n \subset \Lambda^{2\pi^2(\frac{\lambda+1}{\lambda})^2-\epsilon} S^n$ for some $\epsilon > 0$ is sufficiently small, where we use notations in Proposition 2.1. Thus by Proposition 2.2, there are n critical values

$$0 < \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{n-1} < 2\pi^2 \left(\frac{\lambda+1}{\lambda}\right)^2. \tag{3.1}$$

of E. Hence we can find n closed geodesics $\{c_k\}_{0 \leq k \leq n-1}$ of (S^n, F) with $E(c_k) = \lambda_k$. Clearly each c_k is a prime closed geodesic without self-intersections, since otherwise we should have

$$E(c_k) \geq \frac{1}{2} L(c_k)^2 \geq 2\pi^2 \left(\frac{\lambda+1}{\lambda}\right)^2. \tag{3.2}$$

Now if $0 < \lambda_0 < \lambda_1 < \cdots < \lambda_{n-1}$, the geodesics $\{c_k\}_{0 \leq k \leq n-1}$ must be distinct since their energies are different. Otherwise by Proposition 2.2, there are infinitely many non-constant closed geodesics below the level set $E^{-1}(\lambda_{n-1} + \epsilon)$. These proves the theorem. ∎

Proof of Theorem 1.3. The theorem follows directly from Theorem 3, 4 of [Rad4] and Theorem 1.2. ∎

Proof of Theorem 1.4. Suppose the condition $\left(\frac{\lambda}{\lambda+1}\right)^2 \leq \delta < K \leq 1$ holds. Then Theorem 3 of [Rad4] implies $L(c_k) \geq \pi(1 + \frac{1}{\lambda})$.

We have the following two cases.

Case 1. *We have $0 < \lambda_0 < \lambda_1 < \cdots < \lambda_{n-1}$.*

If there does not exist $\delta_k > 0$ such that any closed geodesic c with $E(c) \in (\lambda_k - \delta_k, \lambda_k + \delta_k)$ is isolated. Then by the Palais-Smale condition and Proposition 2.4, we can find a degenerate closed geodesic c_k with $E(c_k) = \lambda_k$. In particular, c_k is non-hyperbolic.

Otherwise we can apply Proposition 2.3 to find a closed geodesic c_k satisfying $E(c_k) = \lambda_k$ and $i(c) \leq n - 1 + 2k \leq i(c) + \nu(c)$. Clearly, in order to prove the theorem, it is sufficient to assume c_k is hyperbolic for $0 \leq k \leq n - 1$, then we have $i(c_k) = n - 1 + 2k$.

Choose $p, q \in \mathbf{N}$ such that $\frac{p}{q} < \frac{\lambda+1}{\lambda}\sqrt{\delta}$, then we have $L(c_k^q) \geq q\pi(1 + \frac{1}{\lambda}) > \frac{p\pi}{\sqrt{\delta}}$. Hence by the Morse-Schoenberg comparison theorem(cf. P. 220 of [BTZ1]), we have $i(c_k^q) \geq p(n - 1)$. By Corollary 2.3 of [BTZ1], we have $i(c_k^q) = q(n - 1 + 2k)$. Thus we must have $\frac{p}{q} \leq \frac{n-1+2k}{n-1}$. Thus if $\frac{2n-3}{n-1}\frac{\lambda}{\lambda+1} < \sqrt{\delta} \leq 1$, the geodesics c_k for $0 \leq k < [\frac{n}{2}]$ can't be hyperbolic.

Suppose $F^2 < \frac{g_0}{\rho}$, then by the proof of Proposition2.2, we have $L(c_k) = \sqrt{2E(c_k)} < \frac{2\pi}{\sqrt{\rho}}$. Choose $p, q \in \mathbf{N}$ such that $L(c_k) < \frac{p\pi}{q} < \frac{2\pi}{\sqrt{\rho}}$. Then $L(c_k^q) < p\pi$, and hence $i(c_k^q) + \nu(c_k^q) \leq p(n - 1)$ by

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the Morse-Schoenberg comparison theorem. Since c_k is hyperbolic, we have $i(c_k) = n - 1 + 2k$ and $i(c_k^q) = q(n - 1 + 2k)$. Hence we must have $\frac{p}{q} \geq \frac{n-1+2k}{n-1}$. Thus if $\frac{2(n-1)}{2n-1} < \sqrt{\rho} \leq 1$, the geodesics c_k for $[\frac{n+1}{2}] \leq k \leq n - 1$ can't be hyperbolic.

Hence the theorem holds in this case.

Case 2. *We have $\lambda_k = \lambda_{k+1}$ for some $0 \leq k < n - 1$.*

We claim that there must be infinitely many non-hyperbolic prime closed geodesics on (S^n, F).

Suppose the claim does not hold. Then there are finitely many non-hyperbolic closed geodesics $\{d_1, \ldots, d_l\}$ in $\Lambda^{\lambda_k + 1}$. Choose disjoint S^1-invariant open neighborhoods $\mathcal{U}_1, \ldots, \mathcal{U}_l$ such that $S^1 \cdot d_i \subset \mathcal{U}_i$ and $\mathcal{U}_i$ is S^1-homotopic to $S^1 \cdot d_i$. By shrinking $\mathcal{U}_i$ if necessary, we may assume $\partial \mathcal{U}_i \cap crit(E) = \emptyset$ for each i. Note that there are finitely many non-constant closed geodesics in $\Lambda^{\lambda_k + 1} \setminus (\cup_{1 \leq i \leq l} \mathcal{U}_i)$ by Proposition 2.4. Denote them by $\{d_{l+1}, \ldots, d_m\}$. Choose disjoint S^1-invariant open neighborhoods $\mathcal{U}_{l+1}, \ldots, \mathcal{U}_m$ such that $S^1 \cdot d_i \subset \mathcal{U}_i$ and $\mathcal{U}_i$ is S^1-homotopic to $S^1 \cdot d_i$ for $l + 1 \leq i \leq m$. By shrinking them if necessary, we may assume $\mathcal{U}_i \cap \mathcal{U}_j = \emptyset$ for $1 \leq i, j \leq m$ and $i \neq j$. Denote by $\mathcal{U} = \cup_{1 \leq i \leq m} \mathcal{U}_i$. Then we have $crit(E) \cap E^{-1}(\lambda_k) \subset \mathcal{U}$. Let $i : \mathcal{U} \to \Lambda S^n$ be the inclusion and $\eta \in H_{S^1}^2(\Lambda S^n)$ as in the proof of Proposition 2.2. Then we have $i^*(\eta) \neq 0$ as in Proposition 2.2. But $\mathcal{U}$ is S^1-homotopic to a union of a finite number of disjoint circles. Hence we have $i^*(\eta) = 0$. This contradiction proves there must be infinitely many non-hyperbolic prime closed geodesics on (S^n, F).

Combining the above two cases, we obtain the theorem. ∎

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